Figuring out probability is easy. All you have to do is count the possible outcomes. This week we take a closer look at problems that involve counting things.

This lesson has two parts.

- The first part describes combinations and permutations, and introduces factorials. It explains how to solve problems using advanced calculators.
- The second part explains how a statistics calculator solves the problems. *If you only have a basic calculator, then use these formulas to do the homework.*

Combinations and Permutations

To count the outcomes for computing probabilities, we often need a methodical way to count things. This leads to the concepts of combinations and permutations.

Combinations are groups of things where order is *not* important.
Permutations are different orderings of a group, where the order *is* important.

Last week we talked about "selection without replacement". This becomes our main focus this week, where we examine selecting items from a group.

Sometimes the order of things selected is important: letters forming a word, or who sits where, or who takes a turn in which order. These are *permutations*.

Other times it doesn't matter which order things are selected: socks from a drawer, a hand of cards from a deck, or the Washington State lottery. These are *combinations*.

Permutations

Making permutations of a group is what you do when you find all the different orderings of the group. Examples of permutation problems:

1. Of three people (Ann, Bob and Carol) two are selected to be president and vice-president. How many possible different president/vice-president pairs could be selected?
   Answer: There are 6 different president/vice-president pairs or permutations:

Pres.	Vice Pres.
Ann -	Bob
Bob -	Ann
Ann -	Carol
Carol -	Ann
Bob -	Carol
Carol -	Bob

2. How many different letter orderings can you make out of the word CATS?
   Answer: There are 24 different orderings, or permutations:

CATS
CAST
CTAS
CTSA
CSAT
CSTA
ACTS
ACST
ATCS
ATSC
ASCT
ASTC
TACS
TASC
TCAS
TCSA
TSAC
TSCA
SATC
SACT
STAC
STCA
SCAT
SCTA

3. How many different orderings of the letters in the word MOON are possible?
   Answer: 12 different orderings (permutations) of the letters.

MOON
MONO
MNOO
OMON
OMNO
OOMN
ONMO
ONOM
OONM
NMOO
NOOM
NOMO

Why is there fewer orderings of MOON than CATS?
Because the letter ‘O’ is repeated; there are fewer unique letters to choose from.

Combinations

Combinations are different selections of things where order doesn't count. In this type of problem you are asked for the number of possible selections from a group of things.

Example: A roller coaster has 3 seats, and 4 children (whose names are A, B, C and D) want to ride it. How many ride combinations are possible?

Answer: 4 combinations: ABC, ABD, BCD, CDA

Note that in this problem, it is not important which child gets into which seat. (So I guess these aren't brothers and sisters!) Since order is not important, ABC is the same as CBA and BAC.

Factorials

Factorials are used to compute permutations and combinations. A factorial means "the product of the first *n* whole numbers". It is written with the surprising symbol of an exclamation point: ***n*!**

Examples:
0! = = 1
1! = 1 = 1
2! = 2 x 1 = 2
3! = 3 x 2 x 1 = 6
4! = 4 x 3 x 2 x 1 = 24
5! = 5 x 4 x 3 x 2 x 1 = 120
6! = 6 x 5 x 4 x 3 x 2 x 1 = 720

You can see that factorials grow larger very quickly. Note that by definition, the factorial of zero is defined to be 1.

However, factorial arithmetic can be simple when it occurs in fractions:

$$\frac{6!}{5!} = \frac{6\times5\times4\times3\times2\times1}{5\times4\times3\times21} = 6$$

$$\frac{8!}{5!} = \frac{8\times7\times6\times5\times4\times3\times2\times1}{5\times4\times3\times2\times1} = 8\times7\times6 = 336$$

Factorial arithmetic for addition and subtraction is like normal operations:

$$6!-3! = 720-6 = 714$$

$$(6-3)! = 3! = 6$$

Using Your Calculator for Combinations and Permutations

If your calculator has statistical functions, then it has functions to compute factorials and combinations and permutations. The *TI Math Explorer Plus* and many other advanced calculators can do it.

- To find permutations of n things where r are the same, press: **n 2^{nd} [nPr] r =**
- To find combinations of n things taken r at a time, press: **n 2^{nd} [nCr] r =**
- To find factorial of n press: **n 2^{nd} [x!]**

In each case, your calculator is computing an expression that is described later. Using these calculators can be a great help during, say, a math contest!

The general method of solving these types of problems is to reword the question as follows and then pressing the keys for either [nPr] or [nCr]:

“Find the __________ of _______ things taken _______ at a time.”
combinations *number n* *number r*
permutations

Example: A baseball team has 13 members. How many lineups of 9 players are possible? The position of each member in a lineup is not important.

Answer: Since the order is not important, this must be a combination problem (not a permutation). This problem can be re-worded as “find the combinations of 13 things taken 9 at a time”.

Press the keys: 13 2^{nd} [nCr] 9 =

And the display shows 715.

How Your Calculator Works

Don’t have a calculator? Want to do it the hard way? Or you just have a simple calculator? Then here’s how to do it all yourself!

How to Compute Permutations and Combinations

Fortunately, with the help of the factorial you don't have to write down all the possible permutations and combinations to count them. For problems using large numbers of permutations, counting them by hand is almost impossible.

If I asked how many ways we could arrange the letters in WONDERFUL, you could work for weeks and still not get them all. So, here are the equations for each type of permutation or combination.

1. Number of permutations of n different things: $P = n!$

Example: How many permutations of the letters in the word WONDERFUL are possible?

Answer: No letters are repeated. There are nine ways to choose the first letter, and eight ways to choose the second letter, then seven ways for the next, and so forth:

$$P = 9! = 9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 362{,}880$$

Example: How many different orderings of the letters in the word CATS are possible?

Answer: $P = 4! = 4 \times 3 \times 2 \times 1 = 24$ (Compare with earlier.)

2. Number of permutations of n things where r things are the same: $P = \frac{n!}{r!}$

Example: How many orderings of the letters in the word MOON are possible?

Answer:

$$P = \frac{4!}{2!} = \frac{4 \times 3 \times 2 \times 1}{2 \times 1} = 4 \times 3 = 12$$

Example: How many orderings of MISSISSIPPI are possible?

Answer: There are eleven letters. But that ‘I’ and ‘S’ are repeated four times, and ‘P’ is repeated twice.

$$P = \frac{11!}{4! \times 4! \times 2!} = 34650$$

3. Number of permutations of n things taken r at a time: ${}_nP_r = \frac{n!}{(n-r)!}$

Example: Out of three people (Ann, Bob and Carol) two are elected to president and vice-president. How many pairs can be selected?

Answer: This is the number of permutations of 3 things taken 2 at a time:

$$P = \frac{3!}{(3-2)!} = \frac{3 \times 2 \times 1}{1} = 6$$

(Compare with earlier)

Example: The four kids in a family are arguing over who sits where in their family car which has four passenger seats. How many possible seating arrangements are there?

Answer: Order is important, so this the number of permutations of 4 people taken 4 at a time:

$$P = \frac{4!}{(4-4)!} = \frac{4\times 3\times 2\times 1}{0!} = \frac{24}{1} = 24$$

4. When order doesn't count, the number of combinations of *n* things taken *r* at a time: $${}_nC_r = \frac{n!}{r!\times(n-r)!}$$

Example: A roller coaster has 3 seats and 4 children want to ride. How many ride combinations are possible?

Answer: There are four children selected three at a time, so $n = 4$ and $m = 3$

$$C = \frac{4!}{3!\times(4-3)!} = \frac{4!}{3!\times 1!} = \frac{4\times 3\times 2\times 1}{3\times 2\times 1} = 4$$ (Compare to earlier.)

Example: A baseball team has 13 members. How many lineups of 9 players are possible? The position of each member in the lineup is not important.

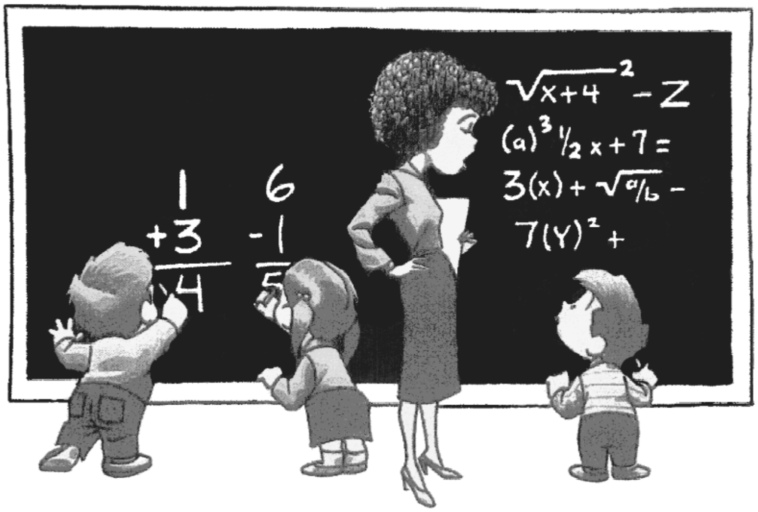
Answer: $$C = \frac{13!}{9!\times(13-9)!} = \frac{13!}{(9!)\times(4!)}$$

$$C = \frac{13\times 12\times 11\times 10\times 9\times 8\times 7\times 6\times 5\times 4\times 3\times 2\times 1}{(9\times 8\times 7\times 6\times 5\times 4\times 3\times 2\times 1)\times(4\times 3\times 2\times 1)}$$

$$C = \frac{13\times 12\times 11\times 10}{4\times 3\times 2\times 1}$$

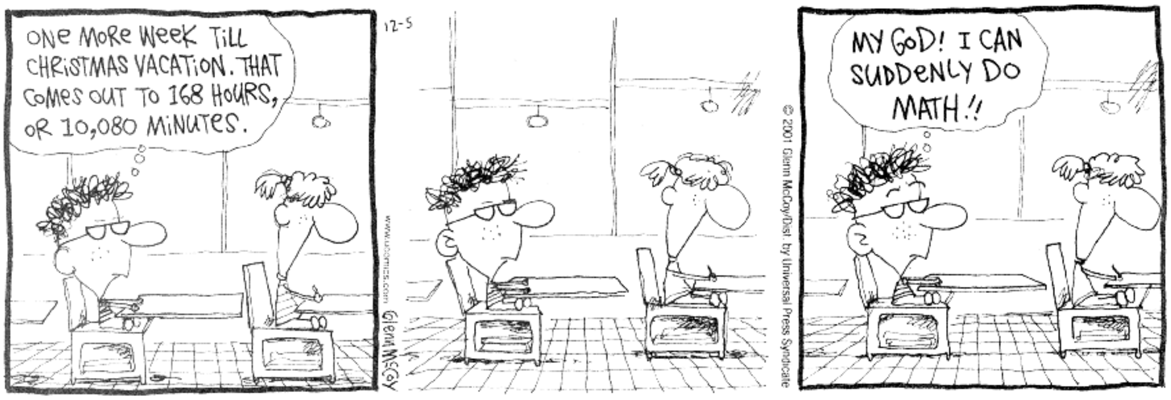
$$C = \frac{17160}{24} = 715$$

As you can see, answering this problem without the use of factorials and the formulas for combinations would be very difficult!

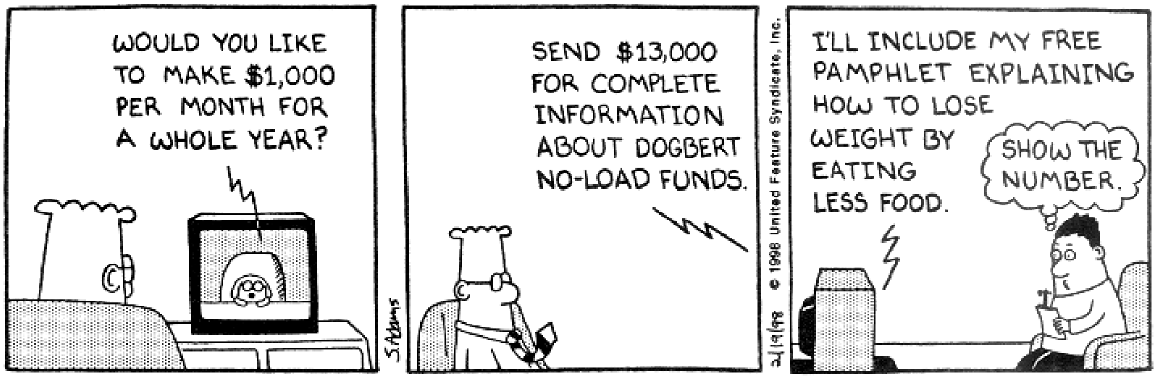


"How many times do I have to tell you...
you're not supposed to read ahead."

The Duplex – Glenn McCoy



Dilbert – Scott Adams



Permutations of Problems

1) Compute these single factorials:

a) 1! =

b) (5 - 3)! =

c) 5! - 3! =

d) 5! =

e) 3! =

f) 7! =

g) 7! - 5! =

h) (8 - 6)! =

i) *Extra credit!* 0! =

2) Compute these quotients of factorials:

a) $\frac{7!}{6!} =$

b) $\frac{10!}{8!} =$

c) $\frac{88!}{86!} =$

3) *Extra credit:* What is the largest number for which your calculator can show the factorial? (*Hint:* It is less than 100.) You can work this out, even if your calculator has only the basic functions.

4) Compute these combinations of things, where order is not important.

a) Find the combinations of five items taken two at a time.

$_5C_2$ =

b) Find the combinations of six things taken three at a time.

$_6C_3$ =

c) Suppose you take all the members of a group together.
Find the combinations of five things taken five at a time.

$_5C_5$ =

d) Find the combinations of nine things taken eight at a time.

$_9C_8$ =

5) Compute the number of permutations (order is important).

a) Find the permutations of four things taken two at a time.

$_4P_2 =$

b) Find the permutations of five things taken three at a time.

$_5P_3 =$

c) Suppose you take all the members of a group together.
Find the permutations of four things taken four at a time.

$_4P_4 =$

d) Find the permutations of seven things taken two at a time.

$_7P_2 =$

e) Find the permutations of seven things taken three at a time.

$_7P_3 =$

6) Compute the number of permutations.

a) How many orderings of the letters in the word SMILE are possible?

b) How many ways can 4 letters out of the word WONDERFUL be ordered?
For example, WOND, WONE, WONR, …
Hint: Just say how many, don't list them.

7) After you did last week's homework, the nasty Pickled Porpoise learned something about the three-digit security code of your computer controlled coilgun protection system around your bedroom. (http://www.oz.net/~coilgun) His evil but stupid henchmen determined the digits are 4, 2, and 5 (at considerable difficulty), but they don't know the order of the digits. What is the probability of guessing the right code on their first random trial?

8) *Extra credit:* Eight people met at a New Year's Eve party and all shake hands. How many handshakes were there?
(*Hint:* It takes two people to shake hands and order doesn't count.)

9) Mental Math: do these in your head, and write down the answers.
Leave all answers as reduced fractions, and in terms of radicals and pi.

a) What is your name?

b) What is (-3) cubed?

c) Compute $\frac{3/8}{2/8}$

d) What is 6 divided by -2?

e) What is 7 – (–3)?

f) What is 5 plus –2 ?

g) What is 5 times 5 times –5 ?

h) What is +2 times –2?

i) Work backward to solve this problem:

The Backward Boy loves to sleep during the daytime.
He snored for 4 hours longer than he talked in his sleep.
He talked in his sleep for twice as long as he walked in his sleep.
He walked in his sleep for 1 hour.
For how many hours did the Backward Boy snore?

Did you check your work? It’s okay to use a calculator for checking results.

You’re done! Detach the homework from the lesson, and turn in just the *homework*. And did you know that 37.4% of all statistics are made up on the spot?